3 Inductive Proofs and an Alternative Proof for Theorem: n odd implies n^2 odd

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Assumed number-theoretic properties:

Notation:
- $\mathbb{Z}$ is the set of integers (historically, Zis for ?zahlen?, the German word for ?numbers?.
- $\mathbb{Z}^+$ is the set of non-negative integers (0, 1, 2, . . .)

Definitions:
- An integer n is even if it is evenly divisible by 2; that is, $kn = 2k$ for some $k \in \mathbb{Z}$
- An integer n is odd if it is not evenly divisible by 2; that is, $kn = 2k + 1$ for some $k \in \mathbb{Z}$

Properties/Theorems: Suppose r and s are integers.
- if r is even, then $r - 2$ is even and $r - 1$ is odd.
- Ifr is odd, then $r - 2$ is odd and $r - 1$ is even.
- If r and s are both even or both odd, then $r + s$ is even.
- If one of r and s is odd and the other is even, then $r + s$ is odd.

Theorem: If $n \in \mathbb{Z}^+$ is odd, then n^2 also is odd.

Proof 1: by induction on n, using the following
Induction Hypothesis IH(n): If $n \in \mathbb{Z}$, $k \leq n$ is odd, then k^2 is odd.

Base cases ($n = 0$ and $n = 1$): Both 1 and 1^2 are trivially odd.
- Since 0 is even, the "if" statement is false, and IH(0) holds.
- Both 1 and 1^2 are trivially odd, so IH(1) holds.

Inductive step: For $n > 1$, we show IH(n) being true implies IH($n + 1$) is true.
Proof of inductive step: Consider $n + 1$
If $n + 1$ is even, the IH($n + 1$) makes no claims about $k = n + 1$, and for all odd $k < n + 1$, k^2 is odd by IH(n).
If $n + 1$ is odd with $n > 1$, then
- For $k < n + 1$, then k odd implies k^2 odd by IH(n).
- For $k = n + 1$,
  - § if k even, then IH($n + 1$) makes no claims beyond IH(n).
  - § if $k = n + 1$ odd, then $n - 1$ is odd and at least 1.
    Since $k = n - 1$ is odd, $(n - 1)^2$ is odd by IH(n).
    Algebraically, $(n + 1)^2 = ((n - 1) + 2)^2$
    $= (n - 1)^2 + 4(n - 1) + 4$
    Altogether, $(n + 1)$ is the sum of an odd and two even integers, and thus is odd.

This completes the inductive step.

Proof 2: by induction, using the following

Induction Hypothesis IH(n): For k equal either n or $n-1$, then

- If $k \in \mathbb{Z}^+$ is odd, then k^2 is odd.
- If $k \in \mathbb{Z}^+$ is even, then k^2 is even.

Base cases ($n = 0$ and $n = 1$): $0^2 = 0$ are both even, and $1^2 = 1$ are both odd.

Inductive step: For $n > 1$, we show that if IH(n) is true, then IH($n+1$) is true.

Proof of inductive step: For $k < n+1$, the claims are true by IH(n), so consider $k = n+1$

Algebraically,
$$(n+1)^2 = ((n-1)+2)^2$$
$$= (n-1)^2 + 4(n-1) + 4$$

Also, for any integer n, $4(n-1)+4$ is even.

Putting these pieces together,

- If $n+1$ is odd with $n > 1$, then $n-1$ is odd, and $(n-1)^2$ is odd by IH(n)
  Thus, $(n+1)^2 = (n-1)^2 + 4(n-1) + 4$ is an odd plus an even, and so is odd.
- If $n+1$ is even with $n > 1$, then $n-1$ is even, and $(n-1)^2$ is even by IH(n)
  Thus, $(n+1)^2 = (n-1)^2 + 4(n-1) + 4$ is an even plus an even, and so is even.

In either case, IH($n+1$) holds.

Proof 3: by induction, using the following

Induction Hypothesis IH(n):

- If $n \in \mathbb{Z}^+$ is odd, then n^2 is odd.
- If $n \in \mathbb{Z}^+$ is even, then n^2 is even.

Base case ($n = 0$): $0^2 = 0$ are both even.

Inductive step: For $n > 0$, we show IH(n) being true implies IH($n+1$) is true.

Proof of inductive step: Consider $n+1$

Algebraically, $(n+1)^2 = n^2 + 2n + 1$

Also, for any integer n, $2n+1$ is odd.

Putting these pieces together,

- If $n+1$ is odd with $n > 1$, then n is even, and IH(n) applies.
  Thus, $(n+1)^2 = n^2 + 2n + 1$ is an even plus an odd, and so is odd.
- If $n+1$ is even with $n > 1$, then n is odd, and IH(n) applies..
  Thus, $(n+1)^2 = n^2 + 2n + 1$ is an odd plus an odd, and so is even.

In either case, IH($n+1$) is true.

An Alternative Proof

Alternative Proof: We prove $(2m+1)^2$ is odd for all $m \in \mathbb{Z}^+$.

Induction Hypothesis IH(m): $(2m+1)^2$ is odd.

Base case ($m = 0$): When $m = 0$, $2m+1$ is 1, and $1^2 = 1$ is odd.

Inductive Step: For $m > 0$ we show IH(m) implies IH($m+1$).

> Proof of the inductive step.
> $(2m+1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$
> Since $2(2m^2 + 2m) + 1$ has the form 2 times an integer + 1, we conclude this number (or equivalently $(2m+1)^2$) is odd, establishing IH($m+1$).